Milnor Excision and Beauville–Laszlo Gluing

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This is expository work. No results or arguments presented are claimed to be original.

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1 Introduction and Background

1.1 A Krash Course on K-Theory

K-theory refers to the study of functors K_i which associate to a ring Λ a sequence of abelian groups $K_i\Lambda$. The definition of the higher K-groups, K_i for $i \geq 2$, eluded mathematicians until the 1970s. Until then, the only K-groups one could hope to study were K_0, K_1, and K_2, the last of which did not have a universally accepted definition at the time. These groups are related to each other by an exact sequence:

$$K_2\mathfrak{a} \to K_2(\Lambda) \to K_2(\Lambda/\mathfrak{a}) \to K_1\mathfrak{a} \to K_1(\Lambda) \to K_1(\Lambda/\mathfrak{a}) \to K_0\mathfrak{a} \to K_0(\Lambda) \to K_0(\Lambda/\mathfrak{a})$$

where $\mathfrak{a}$ is an ideal in Λ. It is not clear that we have a well-formed definition of $K_i\mathfrak{a}$, and in fact, it turns out that for $i > 0$, $K_i\mathfrak{a}$ depends on the ambient ring. In this section, we will define only K_0 and K_1, as these are the only groups needed to motivate the subject of the paper. Milnor defines these in [1]:

Definition 1.1. *Consider the category of finitely generated projective modules over a ring Λ. The projective module group $K_0\Lambda$ is the abelian group generated by isomorphism classes $[P]$ of finitely generated projective modules over Λ subject to the relation that $[P]+[Q] = [P \oplus Q]$.*

Definition 1.2. *Let $GL(\Lambda)$ denote the direct limit of the sequence $GL_1(\Lambda) \subset GL_2(\Lambda) \subset GL_3(\Lambda) \subset \ldots$. A matrix in $GL(\Lambda)$ is elementary if it exactly resembles the identity matrix except for one off-diagonal entry. The group generated by elementary matrices is a normal subgroup of $GL(\Lambda)$ which we denote $E(\Lambda)$. We define the Whitehead group $K_1\Lambda$ as the quotient group $GL(\Lambda)/E(\Lambda)$.*

Remark 1.3. *In practice, the groups $K_0\Lambda$ and $K_1\Lambda$ are difficult to compute. However, one might notice that the exact sequence above resembles the exact sequence used to relate the homology groups of a space, a subspace, and its corresponding quotient space. This instills hope that we can use some tools from classical algebraic topology to aid the study of K-groups.*

1.2 Excision and the Mayer-Vietoris Sequence

Recall that the excision theorem describes when we can remove a subspace $Z \subset A$ so that the relative homology groups $H_n(X, A)$ and $H_n(X \setminus Z, A \setminus Z)$ are isomorphic. In [2], Hatcher describes Excision and the Mayer-Vietoris exact sequence:

Theorem 1.4 (Excision). *Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$ for all n.*

The proof of the excision theorem is lengthy and beyond the scope of the paper. In general, we would like to be able to compute the homology of a space by breaking it up into simpler pieces. Excision allows us to do exactly that:

Corollary 1.5. *For subspaces $A, B \subset X$ whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$*

Proof. Set $B = X \setminus Z$ and $Z = X \setminus B$ so that $A \cap B = A \setminus Z$. Then, since $X \setminus int(B)$ is the closure of Z, the condition that the closure of Z be in the interior of A is equivalent to requiring that $X = int(A) \cup int(B)$. $\square$

Remark 1.6. *This equivalent characterization of excision gives rise to the Mayer-Vietoris sequence, an incredibly powerful computational tool that gives us information about the homology of a space X in terms of the homology of A, B, and $A \cap B$. In the next section, we will see that its K-theoretic analog, Milnor excision, is similarly useful.*

Definition 1.7 (Mayer-Vietoris sequence). *Given subspaces $A, B \subset X$ as above, the corresponding Mayer-Vietoris sequence is the long exact sequence:*

$$\ldots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \ldots \to H_0(X) \to 0$$

1.3 The f-adic Completion of a Ring

In the following section, we will state the excision theorem for rings and schemes. For the rest of this section, we will give the background necessary to understand the analogous statement for the gluing of rings and sheaves: the Beauville–Laszlo theorem. [3] gives the following definitions:

Definition 1.8. *Let A be a commutative ring and let $\mathfrak{a}$ be a finitely generated ideal of A. The filtration $(\mathfrak{a}^n A)_n$ defines a topology on A called the $\mathfrak{a}$-adic topology. This is the topology with a basis of open sets $\{a + \mathfrak{a}^n A : a \in A, n \in \mathbb{Z}_{>0}\}$.*

Definition 1.9. *The $\mathfrak{a}$-adic completion of A is $\hat{A}_{\mathfrak{a}} := \varprojlim A/\mathfrak{a}^n A$. When the ideal in question is unambiguous, we will simply write $\hat{A}$.*

One checks that the inverse limit exists and retains a ring structure.

Remark 1.10. *We are primarily interested in the case where $\mathfrak{a} = (f)$ for some $f \in A$, and we will usually write f-adic rather than (f)-adic for brevity. It may not be immediately clear that $\hat{A}$ is actually complete with respect to the $\mathfrak{a}$-adic topology. Fortunately, this is the case.*

Proposition 1.11 ([4])**.** *$\hat{A}$ is complete with respect to the $\mathfrak{a}$-adic topology. In particular, the map $i : A \to \hat{A}$ is an isomorphism if and only if A is complete in this topology.*

Proof. Define a filtration $(\hat{A}_n)_n$ on $\hat{A}$ by $\hat{A}_n := \{(a_m)_m \in \hat{A} : a_m = 0 \text{ for all } m \leq n\}$. Let $\{a_k\}_k$ be a Cauchy sequence in $\hat{A}$. For all $n > 0$ there exists k_n so that for all $k \geq k_n$, $a_k - a_{k_n} \in \mathfrak{a}^n$. We can write $a_k = (a_{k,1}, a_{k,2}, ...)$ and we see that this implies $a_{k_n,i} = a_{k,i}$ when $i \leq n$ and $k \geq k_n$. Then the limit of $\{a_k\}_k$ is $\hat{a} = (a_{k_1,1}, a_{k_2,2}, ...) \in \hat{A}$.

Suppose i is an isomorphism and let $\{a_i\}_i$ be a Cauchy sequence that converges to $\hat{a} \in \hat{A}$. Then $\hat{a}$ has a preimage in A that the preimage of $\{a_i\}_i$ must converge to. Since all sequences in A have an image in $\hat{A}$, this implies A is complete.

Conversely, suppose A is complete and let $\{a_i\}_i$ be a Cauchy sequence in $\hat{A}$ which converges to $\hat{a} \in \hat{A}$. Then for all $n > 0$, $a_{n+1} - a_n \in \mathfrak{a}^n$ so the sequence is Cauchy in A. Since A is complete, there is some $a \in A$ this sequence converges to that must map to $\hat{a}$. Now, if $\{a_i\}_i$ is a Cauchy sequence in $\hat{A}$ which converges to $0 \in \hat{A}$. Then its preimage must converge to 0 in A, so the kernel of i is trivial. $\square$

Remark 1.12. *These definitions work equally well if we replace A with an A-module M. If M is finitely generated and A is noetherian then $\hat{M}$ is also complete in the $\mathfrak{a}$-adic topology.*

1.4 The Tor Functors

In a later section, we will use the Tor functors to prove the Beauville-Laszlo theorem. We briefly recall their definitions, given in [5], here. Given a commutative ring A, we have a bifunctor $(-) \otimes_A (-) : A\text{-Mod} \times A\text{-Mod} \to \mathrm{Ab}$. The functors Tor_i^A are the derived functors of $\otimes_A$.

Definition 1.13. *Consider an A-module N. Given a projective resolution $\mathcal{P}$ of another A-module M:*

$$... \to \mathcal{P}_2 \to \mathcal{P}_1 \to \mathcal{P}_0 \to M \to 0$$

define $\mathcal{P} \otimes N$ to be

$$... \to \mathcal{P}_2 \otimes N \to \mathcal{P}_1 \otimes N \to \mathcal{P}_0 \otimes N \to M \otimes N \to 0$$

Definition 1.14. *Given A-modules M and N, we define $\mathrm{Tor}_i^A(M, N) := H_i(\mathcal{P} \otimes N)$. This defines bifunctors $\mathrm{Tor}_i^A(-,-) : A\text{-Mod} \times A\text{-Mod} \to \mathrm{Ab}$. When the underlying ring is clear, we will write only Tor_i.*

Remark 1.15. *$\mathrm{Tor}_i^A(M, N)$ is independent of the choice of projective resolution $\mathcal{P}$ and is thus well-defined. Given two projective resolutions $\mathcal{P}$ and $\mathcal{Q}$ of M, one can prove this by constructing a map of chain complexes $\mathcal{P} \to \mathcal{Q}$ and showing it is unique up to chain homotopy. This yields an isomorphism of homology groups $H_i(\mathcal{Q} \otimes N) \cong H_i(\mathcal{P} \otimes N)$.*

2 Milnor Excision

Most claims and arguments made in this section are Milnor's, presented in [1].

2.1 Examples: $K_0\Lambda$ of Local Rings and Dedekind Domains

Remark 2.1. *In the general case, $K_0\Lambda$ is difficult to compute. However, in certain special cases, such as when Λ is a local ring or a Dedekind domain, $K_0\Lambda$ is better understood. We will begin with these examples. This phenomenon closely resembles the motivation for excision in classical algebraic topology, so it is reasonable to want to seek out similar exact sequences in K-theory.*

Theorem 2.2. *All finitely generated projective modules over a local ring Λ are free*

Proof. Let P be such a module and Q such that $P \oplus Q \cong \Lambda^r$. Let $\mathfrak{m}$ be the maximal ideal of Λ and consider $P/\mathfrak{m}P$ and $Q/\mathfrak{m}Q$. These are finite dimensional vector spaces over the field $\Lambda/\mathfrak{m}\Lambda$, so they have bases. Choose a representative for each basis element. Recall that a matrix with entries in a local ring Λ is non-singular if and only if its image in $\Lambda/\mathfrak{m}\Lambda$ is as well. This implies that these representatives form a basis of $P \oplus Q$. This now implies that the representatives from $P/\mathfrak{m}P$ are a basis of P. □

Corollary 2.3. *If Λ is a local ring, $K_0\Lambda$ is the free group on one generator.*

Definition 2.4. *Two non-zero ideals $\mathfrak{a}$ and $\mathfrak{b}$ of a Dedekind domain Λ are said to be in the same ideal class if there exist $x, y \in \Lambda$ so that $x\mathfrak{a} = y\mathfrak{b}$. The ideal class group $C(\Lambda)$ of Λ is the abelian group of ideal classes under multiplication (whose identity is the class of principal ideals). We denote the ideal class of $\mathfrak{a}$ by $\{\mathfrak{a}\}$.*

Theorem 2.5. *If Λ is a Dedekind domain, $K_0\Lambda \cong \mathbb{Z} \oplus C(\Lambda)$ and this isomorphism is canonical. Moreover, this is equipped with a product structure defined by the fact that the product of two elements in $C(\Lambda)$ is 0.*

Before proving 2.5, we recall some basic facts about Dedekind domains and their ideals:

Lemma 2.6.

1. *Every ideal in a Dedekind domain* Λ *is finitely generated and projective and every finitely generated projective module over* Λ *is isomorphic to a finite direct sum of ideals.*

2. *Two direct sums of nonzero ideals* $\mathfrak{a}_1 \oplus ... \oplus \mathfrak{a}_k$ *and* $\mathfrak{b}_1 \oplus ... \oplus \mathfrak{b}_s$ *are isomorphic as* Λ*-modules if and only if* $r = s$ *and* $\{\mathfrak{a}_1 ... \mathfrak{a}_k\} = \{\mathfrak{b}_1 ... \mathfrak{b}_s\}$.

3. *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are nonzero ideals in* Λ*,* $\mathfrak{a} \oplus \mathfrak{b}$ *is isomorphic to* $\Lambda^1 \oplus (\mathfrak{a}\mathfrak{b})$

Proof of 2.5. It is not difficult to check that the map $[\mathfrak{a}_1 \oplus ... \oplus \mathfrak{a}_k] \to (r, \{\mathfrak{a}_1 ... \mathfrak{a}_k\})$ is an isomorphism of groups. Now, let k be a field and consider a morphism $\Lambda \to k$. Note that such a morphism always exists when Λ is commutative. This induces a morphism $K_0\Lambda \to K_0 k \cong \mathbb{Z}$ whose kernel we denote $\tilde{K}_0\Lambda$. Note $\tilde{K}_0\Lambda$ is isomorphic to $C(\Lambda)$ since all modules over k are free (which again illustrates the isomorphism of the underlying groups).

To show the product structure, notice that by 2.6.1 and 2.6.2, we can describe $\tilde{K}_0\Lambda$ as the set of differences of projective Λ-modules $[P] - [Q]$ where $rank(P) = rank(Q)$. Let us take a brief detour to define the rank of a projective module over a Dedekind domain:

The rank of a projective module P at a prime ideal $\mathfrak{p}$ is defined as the rank of the free module $P_\mathfrak{p}$. An application of theorem 2.2 shows that this is free since $\Lambda_\mathfrak{p}$ is a local ring. One can show that when Λ is a domain, this is independent of the choice of $\mathfrak{p}$, so $rank(P)$ is well defined.

In particular, we can write each element of $\tilde{K}_0\Lambda$ as $[\mathfrak{a}] - [\Lambda^1]$. It now suffices to show that $([a] - [\Lambda^1])([b] - [\Lambda^1]) = 0$. But this follows immediately from 2.5.3. ∎

2.2 Building Finitely Generated Projective Modules

In the case of topological spaces, the excision theorem allowed us to study the homology of a space by covering it with simpler spaces and studying the homology of those spaces. In this section, we will show that one can study the category of finitely generated projective modules over a ring Λ, denoted Proj(Λ), by studying finitely generated projective modules over Λ_1 and Λ_2 whose groups of projective modules may be simpler. We will do this by showing that when the map $j_2 : \Lambda_2 \to \Lambda$ is surjective, the pullback square of rings:

$$\begin{tikzcd}
\Lambda \arrow[r, "i_1"] \arrow[d, "i_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \Lambda_1 \arrow[d, "j_1"] \\
\Lambda_2 \arrow[r, "j_2"'] & \Lambda'
\end{tikzcd}$$

induces a pullback square of categories:

$$\begin{tikzcd}
\mathrm{Proj}(\Lambda) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Proj}(\Lambda_1) \arrow[d] \\
\mathrm{Proj}(\Lambda_2) \arrow[r] & \mathrm{Proj}(\Lambda')
\end{tikzcd}$$

Recall that given a Λ-module P and a ring homomorphism $f : \Lambda \to \Lambda'$, we have an induced Λ' module, $\Lambda' \otimes_\Lambda P$ which we denote $f_\# P$. This gives rise to a Λ-linear map $f_* : P \to f_\# P$. Consider finitely generated projective modules P_1 and P_2 over Λ_1 and Λ_2, respectively.

Theorem 2.7 (**Milnor Excision**). *Suppose we have an isomorphism of Λ'-modules $h : j_{1\#}P_1 \to j_{2\#}P_2$. Let $M = M(P_1, P_2, h)$ be the fiber product of P_1 and P_2 over $j_{2\#}P_2$. Recall that this can be viewed as the set of pairs $(p_1, p_2) \in P_1 \times P_2$ where $hj_{1*}(p_1) = j_{2*}(p_2)$. Define a Λ-module structure on M by $\lambda \cdot (p_1, p_2) := (i_1(\lambda)p_1, i_2(\lambda)p_2)$ where $\lambda \in \Lambda$ and $(p_1, p_2) \in M$. Then:*

1. *M is finitely generated and projective over Λ.*

2. *Every finitely generated projective Λ-module is isomorphic to $M(\Lambda_1, \Lambda_2, h)$ for suitably chosen Λ_1, Λ_2, and h.*

3. *$P_1 \cong i_{1\#}M$ and $P_2 \cong i_{2\#}M$.*

To prove the above, we first consider the special case where P_1 and P_2 are free modules. Choose bases $\{x_\alpha\}_\alpha$ and $\{y_\beta\}_\beta$ for P_1 and P_2. These determine bases $\{j_{1*}x_\alpha\}_\alpha$ of $j_{1\#}P_1$ and $\{j_{2*}y_\beta\}_\beta$ of $j_{2\#}P_2$ so that the isomorphism h can be expressed with respect to these bases as an invertible matrix $A = (a_{\alpha,\beta})$ over Λ'.

Lemma 2.8. *Suppose that P_1 and P_2 are free modules and that A is the image under j_2 of an invertible matrix $C = (c_{\alpha,\beta})$ over Λ_2 (so that $a_{\alpha,\beta} = j_2 c_{\alpha,\beta}$). Then M is free and finitely generated.*

Proof. Fix a new basis of P_2 $\{y'_\alpha\}$ by

$$y'_\alpha = \sum_\beta c_{\alpha,\beta} y_\beta$$

We then have that

$$h(j_{1*}x_\alpha) = \sum_\beta a_{\alpha,\beta} j_{2*} y_\beta$$

which we see is exactly the image of $\{y'_\alpha\}$ under j_{2*}. Having now shown that $h(j_{1*}x_\alpha) = j_{2*}y'_\alpha$, we have shown that (x_α, y'_α) is an element of $M(P_1, P_2, h)$ for each α. The set $\{(x_\alpha, y'_\alpha)\}_\alpha$ is therefore a finite basis of M over Λ, so M is free and finitely generated. $\square$

Lemma 2.9. *Suppose that P_1 and P_2 are finitely generated free modules and that j_2 is surjective. Then M is finitely generated and projective.*

Proof. Let Q_1 be a free Λ_1-module such that $rank_{\Lambda_1}(Q_1) = rank_{\Lambda_2}(P_2)$. Similarly, let Q_2 be a free Λ_2-module such that $rank_{\Lambda_2}(Q_2) = rank_{\Lambda_1}(P_1)$. The isomorphism h gives rise to an isomorphism $g : j_{1\#}Q_1 \to j_{2\#}Q_2$ with matrix A^{-1}. Using the definition of M, one checks that $M(P_1, p_2, h) \oplus M(Q_1, Q_2, g) \cong M(P_1 \oplus Q_1, P_2 \oplus Q_2, h \oplus g)$. The matrix $A \oplus A^{-1}$ represents the map $h \oplus g$. We can decompose this matrix as

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -A^{-1} & I \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

where each I is the identity matrix of appropriate size. Since j_2 is surjective, each matrix on the right-hand side is the image of a matrix over Λ_2. The preimage of the left-most matrix on the right-hand side must be of the form

$$\begin{pmatrix} I & * \\ 0 & I \end{pmatrix}$$

which is necessarily invertible. One can show similarly that the other three matrices on the right-hand side are also invertible. Thus, $A \oplus A^{-1}$ is the image under j_2 of an invertible matrix. By lemma 2.8, this implies that $M(P_1 \oplus Q_1, P_2 \oplus Q_2, h \oplus g) \cong M(P_1, P_2, h) \oplus M(Q_1, Q_2, g)$ is a finitely generated free module. This immediately implies $M(P_1, P_2, h)$ is finitely generated and projective. □

Lemma 2.10. *Suppose P_1 and P_2 are projective, then there exist finitely generated modules Q_1 and Q_2 so that $P_1 \oplus Q_1$ and $P_2 \oplus Q_2$ are free and $j_{1\#}Q_1 \cong j_{2\#}Q_2$.*

Proof. Pick N_1 and N_2 so that $P_1 \oplus N_1 \cong (\Lambda_1)^r$ and $P_2 \oplus N_2 \cong (\Lambda_1)^s$ for some $r, s > 0$. Then $j_{1\#}P_1 \oplus j_{1\#}N_1 \cong (\Lambda')^r$ and $j_{2\#}P_2 \oplus j_{2\#}N_2 \cong (\Lambda')^s$. Recall that $j_{1\#}P_1 \cong j_{2\#}P_2$ and let $P' = j_{1\#}P_1$. Then we have that

$$j_{1\#}N_1 \oplus (\Lambda')^s \cong j_{1\#}N_1 \oplus P' \oplus j_{2\#}N_2 \cong (\Lambda')^r \oplus j_{2\#}N_2$$

Now we may define $Q_1 = N_1 \oplus (\Lambda')^s$ and $Q_2 = N_2 \oplus (\Lambda')^r$ and note that these satisfy the desired conditions. □

The proof of 2.7 falls out of the preceding lemmas:

Proof of theorem 2.7.1. Choose Q_1 and Q_2 as in lemma 2.10 and fix an isomorphism $k : j_{1\#}Q_1 \to j_{2\#}Q_2$. By lemma 2.9 $M(P_1, P_2, h) \oplus M(Q_1, Q_2, k)$ is finitely generated and projective, which implies $M(P_1, P_2, h)$ is also finitely generated and projective. □

Proof of theorem 2.7.2. Let P be a finitely generated projective Λ-module and set $P_1 = i_{1\#}P$ and $P_2 = i_{2\#}P$. Recall from the diagram above that $j_1 i_1 = j_2 i_2$ so we have an isomorphism $h : j_{1\#}P_1 \cong j_{2\#}P_2$ and a corresponding commutative diagram:

$$\begin{tikzcd} P \arrow[r, "i_{1*}"] \arrow[d, "i_{2*}"'] & P_1 \arrow[d, "hj_{1*}"] \\ P_2 \arrow[r, "j_{2*}"'] & j_{2\#}P_2 \end{tikzcd}$$

so that $P = M(P_1, P_2, h)$ □

Proof of theorem 2.6.3. We have a natural Λ-linear map $M \to P_1$ which induces a map $f : i_{1\#}M \to P_1$. If as in lemma 2.9, M and P_1 are free, $i_{1\#}M$ and P_1 are free of the same rank, so they are isomorphic.

Now, recall that as in the proof of lemma 2.9, $M(P_1, P_2, h)$ is a direct summand of a free module $M(P_1 \oplus Q_1, P_2 \oplus Q_2, h \oplus \overline{h})$. We have a natural map $M(Q_1, Q_2, \overline{h}) \to Q_1$ which induces a map $\overline{f}$ as above whose image lies in Q_1. Then $f \oplus \overline{f}$ is an isomorphism which in particular implies that f is as well. □

Remark 2.11. *Having shown that the diagram of categories above is indeed a pullback diagram tells us we can apply the excision theorem to affine schemes. Applying the Spec functor to the pullback square of rings immediately gives a pullback square in the category of schemes. This extends globally without much difficulty (in the case of separated schemes) since we can take an affine open cover and use the fact that finite intersections of affine sets are affine to ensure compatibility.*

2.3 Exact Sequences in K-theory

Definition 2.12. *Suppose we have a pullback square satisfying the assumptions used in section 2.2. The (algebra K-theoretic) Mayer-Vietoris sequence of a ring Λ is the exact sequence*

$$K_1\Lambda \to K_1\Lambda_1 \oplus K_1\Lambda_2 \to K_1\Lambda' \to K_0\Lambda \to K_0\Lambda_1 \oplus K_0\Lambda_2 \to K_0\Lambda'$$

Its name is due to the resemblance of the Mayer-Vietoris sequence in classical algebraic topology. We define the homomorphisms $K_i\Lambda \to K_i\Lambda_1 \oplus K_i\Lambda_2 \to K_i\Lambda'$ by $x \mapsto (i_{1*}x, i_{2*}x)$ and $(y,z) \mapsto j_{1*}y - j_{2*}z$. We define the map $\partial : K_1\Lambda' \to K_0\Lambda$ as follows: represent $x \in K_1\Lambda'$ by a matrix in $GL_n(\Lambda')$. This determines an isomorphism of free Λ'-modules $h : j_{1\#}\Lambda_1^n \to j_{2\#}\Lambda_2^n$. We then set $M = M(\Lambda_1^n, \Lambda_2^n, h)$ and define $\partial(x) = [M] - [\Lambda^1]$. One checks briefly that this is well-defined and that it follows from Milnor excision that the sequence is indeed exact.

Definition 2.13. *Let $\mathfrak{a}$ be a (2-sided) ideal in Λ. Let $D = D(\Lambda, \mathfrak{a})$ denote the subring of $\Lambda \times \Lambda$ composed of elements (λ, λ') such that $\lambda = \lambda'$ mod $\mathfrak{a}$. Let $p_1, p_2 : D \to \Lambda$ be the projection maps. We define $K_i\mathfrak{a} := ker(p_{1*})$ where p_{1*} is the induced map $K_iD \to K_i\Lambda$.*

Theorem 2.14. *The sequence*

$$K_1\mathfrak{a} \to K_1\Lambda \to K_1\Lambda/\mathfrak{a} \to K_0\mathfrak{a} \to K_0\Lambda \to K_0\Lambda/\mathfrak{a}$$

is exact.

Proof. The commutative diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda \arrow[d] \\ \Lambda \arrow[r] & \Lambda/\mathfrak{a} \end{tikzcd}$$

satisfies the hypothesis of Milnor excision and thus defines an exact sequence

$$K_1D \to K_1\Lambda \oplus K_1\Lambda \to K_1\Lambda/\mathfrak{a} \to K_0D \to K_0\Lambda \oplus K_0\Lambda \to K_0\Lambda/\mathfrak{a}$$

from which we extract the maps $p_{2*}|_{ker(p_{1*})} : K_i\mathfrak{a} \to K_i\Lambda$ and $\pi_* : K_i\Lambda \to K_i\Lambda/\mathfrak{a}$ where π is the projection $\Lambda \to \Lambda/\mathfrak{a}$. The sequence is exact by construction. ∎

3 Beauville–Laszlo Formal Gluing

3.1 The Beauville–Laszlo Theorem

In the previous section, we gave a scheme-theoretic analog of the excision theorem. In this section, we would like to give a similar analog of formal gluing. Consider a ring A and let $f \in A$ such that f is not a zero divisor. Let $A_f = A[f^{-1}]$ and let $\hat{A}$ be the completion of A in the f-adic topology. We will show that the diagram:

$$\begin{array}{ccc} M_f(A) & \longrightarrow & A_f\text{-Mod} \\ \downarrow & & \downarrow \\ M_f(\hat{A}) & \longrightarrow & \hat{A}_f\text{-Mod} \end{array}$$

where $M_f(A)$ denotes the category of f-regular A-modules (modules where f is not a zero divisor), is a pullback square of categories. Most claims and arguments made in this subsection are presented in either one or both of [6] and [7].

Remark 3.1. *In the case that A is noetherian, the maps $A \to \hat{A}$ and $A \to A_f$ are flat and one can apply Grothendieck's faithfully flat descent theory. The fact that the map $A \to \hat{A}$ need not be flat when A is not noetherian motivates the main result of this section.*

Theorem 3.2 (**Beauville–Laszlo**). *Let F be an A_f-module and G an f-regular $\hat{A}$-module. Suppose there exists an isomorphism of $\hat{A}_f$-modules $\varphi : \hat{A} \otimes_A F \to G_f$. Then there exists an f-regular A-module M equipped with isomorphisms $\alpha : M_f \to F$ and $\beta : M \otimes_A \hat{A} \to G$. This is a functorial bijection $(F, G, \varphi) \longleftrightarrow (M, \alpha, \beta)$. Moreover, if F and G are finitely generated (resp. flat, resp. projective and finitely generated) then M is as well.*

Remark 3.3. *Given (M, α, β), we easily recover (F, G, φ) by $F = M_f$, $G = \hat{A} \otimes_A M$, and $\varphi : \hat{A} \otimes_A M_f \to (\hat{A} \otimes_A M)_f$ is the usual isomorphism. Therefore, we are most concerned with the identification $(F, G, \varphi) \to (M, \alpha, \beta)$.*

Lemma 3.4. *Suppose such an M exists as claimed. The sequence*

$$0 \to M \to F \to G_f/G \to 0$$

is short exact.

Proof. Let us first compute $\mathrm{Tor}_1(A_f/A, M)$. Observe that $A_f/A \cong \varinjlim A/f^nA$, so it suffices to compute $\mathrm{Tor}_1(A/f^nA, M)$. Since f is not a zero divisor and M is f-regular, the multiplication map $\cdot f^n$ is injective on $A \otimes M$. Using the resulting long exact sequence after tensoring with M, we see that $\mathrm{Tor}_1(A/f^nA, M) = 0$ for all n and therefore that $\mathrm{Tor}_1(A_f/A, M) = 0$. This implies that the short exact sequence

$$0 \to A \to A_f \to A_f/A \to 0$$

remains exact after tensoring with M. One checks that A_f/A is f-adically complete as an A-module, so a brief computation and application of proposition 1.11 imply that $A_f/A \cong \hat{A}_f/\hat{A}$ so that

$$0 \to M \to M_f \to (\hat{A}_f \otimes_A M)/(\hat{A} \otimes_A M) \to 0$$

is exact. We then have an isomorphism of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "id"'] & M_f \arrow[r] \arrow[d, "\alpha"] & \frac{(\hat{A}_f \otimes_A M)}{(\hat{A} \otimes_A M)} \arrow[r] \arrow[d, "\beta"] & 0 \arrow[d] \\ 0 \arrow[r] & M \arrow[r] & F \arrow[r] & G_f/G \arrow[r] & 0 \end{tikzcd}$$

□

Corollary 3.5. *If (M, α, β) exists, it is unique up to isomorphism.*

Proof. Any other triple must also yield an isomorphism with the short exact sequence

$$0 \to M \to F \to G_f/G \to 0$$

as in the above proof. This immediately implies such a triple is isomorphic to (M, α, β). □

Lemma 3.6. *Let $\overline{\varphi} : F \to G_f/G$ be the composition of φ and the projection $\pi : G_f \to G_f/G$. Then $\overline{\varphi}$ is surjective.*

Proof. Set $B = A_f \times \hat{A}$ and notice that $\rho : A \to B$ is faithful, so it suffices to check subjectivity after tensoring with B. We have that

$$A_f \otimes_A (G_f/G) \cong (A_f \otimes_A G_f)/(A_f \otimes_A G) \cong G_f/G_f \cong 0$$

so we have reduced to tensoring with $\hat{A}$. Then $1_{\hat{A}} \otimes \overline{\varphi} : \hat{A} \otimes_A F \to G_f/G$ is the composition $\pi \circ \varphi$ where $\pi : G_f \to G_f/G$ is the usual surjection. The claim follows because φ is an isomorphism, so its composition with π remains surjective. □

Proposition 3.7. *Given a triple (F, G, φ), there exists (M, α, β) as described above. Additionally, if F and G are finitely generated (resp. flat, resp. projective and finitely generated), so is M.*

Proof. Set $M = ker(\overline{\varphi})$ and observe that this gives an exact sequence

$$0 \to M \xrightarrow{i} F \xrightarrow{\overline{\varphi}} G_f/G \to 0$$

where $i : M \to F$ is an inclusion. Let α be the induced map $i_f : M_f \to F$. Recall that $A_f \otimes_A (G_f/G) = 0$, so tensoring the exact sequence with A_f shows that i_f is an isomorphism.

To show the existence of β, we first show that $\mathrm{Tor}_1^A(\hat{A}, G_f/G) = 0$. Since $G_f/G = \varinjlim G/f^nG$, it suffices to show $\mathrm{Tor}_1^A(\hat{A}, G/f^nG) = 0$ for any n. Using the short exact sequence

$$0 \to A \xrightarrow{f^n} A \to A/f^nA \to 0$$

we see that $\mathrm{Tor}_1^A(\hat{A}, A/f^nA) = 0$. A nontrivial computation verifies that we have a surjection $\mathrm{Tor}_1^A(\hat{A}, (A/f^nA)^I) \to \mathrm{Tor}_1^A(\hat{A}, G_f/G)$ and that the former is 0. Having shown this, we get that the following sequence is short exact:

$$0 \to \hat{A} \otimes_A M \xrightarrow{1 \otimes i} \hat{A} \otimes_A F \xrightarrow{\pi \circ \varphi} G_f/G \to 0$$

Then we may pick β to be the composition $\varphi \circ (1 \otimes i) : \hat{A} \otimes_A M \to G$ which is an isomorphism by the above exact sequence. Using the fact that the map $\rho : A \to B$ is faithful, the property of being finitely generated (resp. flat, resp. projective and finitely generated) is preserved. This completes the proof of 3.2. □

3.2 An Example: Vector Bundles

Remark 3.8. *Geometrically, the Beauville-Laszlo theorem states that under certain conditions, we can glue sheaves on an affine scheme over a formal neighborhood of a point. As such, we can build vector bundles on $X = Spec(A)$ from a bundle on $U = Spec(A_f)$ and a bundle on a formal neighborhood $Spec(\hat{A})$ of $X \setminus U$ if they agree on $Spec(\hat{A}_f)$. The geometric intuition is more transparent when we restrict the theorem to projective modules (recall the bijection preserves this property). The original motivation for this theorem was an example of gluing vector bundles which follows as a corollary of the theorem:*

Corollary 3.9 ([6])**.** *Let k be a field, A be a k-algebra. and X be a smooth, connected, algebraic curve over k with p a closed k-rational point on X equipped with a choice of local coordinate z. Let F and G be vector bundles over $X^* = (X - p)$ and the disk $\hat{X} = Spec(\hat{A})$ respectively which agree on a formal neighborhood $\hat{X}^*$. Let $X_R = X \times_k Spec(R)$ and $D_R = Spec(R[[z]])$. There is a functorial bijection between isomorphism classes of rank r vector bundles E on X_R with trivializations τ and σ over X_R^* and D_R respectively and the group $GL_r(R((z)))$.*

Proof. Let us start with the same setup as in the proof of the Beauville-Laszlo theorem. Restricting to the case where F and G are free modules of rank r, the theorem implies that isomorphism classes of triples (M, α, β) are in bijection with $GL_r(\hat{A}_f)$ where α and β are trivializations $A_f^r \to M_f$ and $\hat{A}^r \to \hat{A} \otimes_A M$ respectively. This extends globally by taking an affine cover and applying the theorem to the intersections of each element of the cover.

Recall we set $B = A/fA$ and note it is a formally smooth k-algebra. This gives a morphism $B \to \hat{A}/f\hat{A}$. By formal smoothness, this extends to a k-algebra morphism $B \to \hat{A}$ which gives a map $B[[z]] \to \hat{A}$ where $B[[z]]$ is the formal power series ring. Note that this is non-canonical. We see that this is an isomorphism on the associated graded, so we have that $B[[z]] \cong \hat{A}$ and that $B((z)) \cong \hat{A}_f$. Applying the conclusion from the previous paragraph to the setup in the statement of the corollary now completes the proof. $\square$

Remark 3.10 ([8])**.** *An interesting fact about this example is that it enables an alternate construction of the affine Grassmanian Gr_G of a semisimple algebraic group G over k in terms of G-bundles on a smooth, projective curve X. The crucial step made possible by this theorem is that one can glue vector bundles on a curve X by gluing the trivial bundle on $(X - p)$ and the trivial bundle on a formal neighborhood $Spec(k[[z]])$ of p to construct a vector bundle on X. It turns out that all bundles can be constructed this way, a fact due to the non-trivial result that G-bundles on $(X - p) \times S$ for a scheme S are fpqc-locally trivial on S.*

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